Linear Algebra First Year Notes (MT)

Vectors, Lines, Planes, Multiplication of Vectors

Multiplication of vector and scalar: associative, commutative, and distributive over addition.
↳ $(\lambda\mu)\underline{a} = \lambda(\mu\underline{a}) = \mu(\lambda\underline{a})$, $\lambda(\underline{a}+\underline{b}) = \lambda\underline{a} + \lambda\underline{b}$,
$(\lambda+\mu)\underline{a} = \lambda\underline{a} + \mu\underline{a}$

basis vectors and components: given any 3 vectors $\underline{e}_1, \underline{e}_2, \underline{e}_3$ which are LI (do not all lie in a plane) it is possible to write in 3D space to write any other vector in terms of scalar multiples of them: $\underline{a} = a_1\underline{e}_1 + a_2\underline{e}_2 + a_3\underline{e}_3$. $\underline{e}_1, \underline{e}_2$, and $\underline{e}_3$ are said to form a basis and $\underline{a}$ has been resolved into components a_1, a_2, and a_3.

↳ a basis set must:
- ↳ have as many basis vectors as the number of dimensions (must span the space)
- ↳ no basis vector can be written as a sum of the others (LI)

(aka 'inner product') **Scalar Product:** denoted by $\underline{a}\cdot\underline{b}$ (later $\langle\underline{a},\underline{b}\rangle$)

$\underline{a}\cdot\underline{b} \equiv |\underline{a}||\underline{b}|\cos\theta$ if $\underline{a}\cdot\underline{b} = 0$ then $\underline{a} \perp^r \underline{b}$ (given $\underline{a}, \underline{b} \neq 0$), $|\underline{a}| = \sqrt{\underline{a}\cdot\underline{a}}$

Note: a_x (component of $\underline{a}$ in x-direction) $= \underline{i}\cdot\underline{a}$, $a_y = \underline{a}\cdot\underline{j}$, $a_z = \underline{z}\cdot\underline{a}$

The scalar product is commutative and distributive over addition (associativity doesn't apply)
↳ $\underline{a}\cdot\underline{b} = (\underline{b}\cdot\underline{a})^*$
↳ $\underline{a}\cdot(\underline{b}+\underline{c}) = \underline{a}\cdot\underline{b} + \underline{a}\cdot\underline{c}$

Also note → $(\lambda\underline{a})\cdot\underline{b} = \lambda^*(\underline{a}\cdot\underline{b})$ whereas $\underline{a}\cdot(\lambda\underline{b}) = \lambda(\underline{a}\cdot\underline{b})$

Repeat of section in new notation: $\langle\underline{c},\underline{a}+\underline{b}\rangle = \langle\underline{c},\underline{a}\rangle + \langle\underline{c},\underline{b}\rangle$
$\langle\underline{c},\alpha\underline{a}\rangle = \alpha\langle\underline{c},\underline{a}\rangle$
$\langle\alpha\underline{c},\underline{a}\rangle = \alpha^*\langle\underline{c},\underline{a}\rangle$

In index form:
$\underline{a}\cdot\underline{b} = \langle\underline{a},\underline{b}\rangle = a_i b_i$

very similar $\begin{cases} |\underline{a}| = \sqrt{\langle\underline{a},\underline{a}\rangle} \\ \langle\underline{a},\underline{a}\rangle = 0 \text{ iff } \underline{a}=0, \text{ otherwise } \langle\underline{a},\underline{a}\rangle > 0 \end{cases}$

Vector Product: denoted by $\underline{a}\times\underline{b}$ and defined to be a vector of magnitude $|\underline{a}||\underline{b}|\sin\theta$ in direction $\perp^r$ to both $\underline{a}$ and $\underline{b}$

$\underline{a}\times\underline{b} = |\underline{a}||\underline{b}|\,\hat{\underline{n}}\sin\theta$ if $\underline{a}\times\underline{b} = 0$ then $\underline{a} \parallel^r \underline{b}$ (given $\underline{a},\underline{b} \neq 0$)

Also note $\underline{a}\times\underline{a} = \underline{0}$

The direction of $\hat{\underline{n}}$ can be found with RHR

The vector product is anticommutative, distributive over addition, and non-associative
↳ $(\underline{a}+\underline{b})\times\underline{c} = (\underline{a}\times\underline{c}) + (\underline{b}\times\underline{c})$
$\underline{b}\times\underline{a} = -(\underline{a}\times\underline{b})$
$(\underline{a}\times\underline{b})\times\underline{c} \neq \underline{a}\times(\underline{b}\times\underline{c})$

$\underline{a}\times\underline{b} = (a_y b_z - a_z b_y)\underline{i} + (a_z b_x - a_x b_z)\underline{j} + (a_x b_y - a_y b_x)\underline{k}$

$= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$

In index form: if $\underline{c} = \underline{a}\times\underline{b}$,
then $c_i = \varepsilon_{ijk} a_j b_k$

(normal one)

Scalar Triple Product

Denoted by $\langle \underline{a}, \underline{b}, \underline{c} \rangle$ $\quad \langle \underline{a}, \underline{b}, \underline{c} \rangle = \underline{a} \cdot (\underline{b} \times \underline{c})$

This outputs a scalar = the volume of a parallelepiped whose edges are given by $\underline{a}$, $\underline{b}$, and $\underline{c}$.

If $\underline{a}, \underline{b}, \underline{c}$ are coplanar then $\langle \underline{a}, \underline{b}, \underline{c} \rangle = 0$ (if coplanar)

The scalar triple product is unchanged under cyclic permutation of the vectors.

Other permutations give the negative of the original.

This triple product can also be given by a determinant: $\langle \underline{a}, \underline{b}, \underline{c} \rangle = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}$

Vector Triple Product

By the vector triple product we mean the vector $\underline{a} \times (\underline{b} \times \underline{c})$.

Clearly this is $\perp^r$ to $\underline{a}$ and lies in the $\underline{b}, \underline{c}$ plane. Remember that this is non-associative $[\underline{a} \times (\underline{b} \times \underline{c}) \neq (\underline{a} \times \underline{b}) \times \underline{c}]$

$\hookrightarrow \underline{a} \times (\underline{b} \times \underline{c}) = (\underline{a} \cdot \underline{c})\underline{b} - (\underline{a} \cdot \underline{b})\underline{c}$

$(\underline{a} \times \underline{b}) \times \underline{c} = (\underline{a} \cdot \underline{c})\underline{b} - (\underline{b} \cdot \underline{c})\underline{a}$

$\underline{a} \times (\underline{b} \times \underline{c}) + \underline{b} \times (\underline{c} \times \underline{a}) + \underline{c} \times (\underline{a} \times \underline{b}) = \underline{0}$

Equations of Lines, planes, spheres

Equation of a line: $\underline{r} = \underline{a} + \lambda \underline{b}$ or $(\underline{r} - \underline{a}) \times \underline{b} = \underline{0}$

$$\frac{x - a_x}{b_x} = \frac{y - a_y}{b_y} = \frac{z - a_z}{b_z} = \lambda$$

Equation of a plane: $(\underline{r} - \underline{a}) \cdot \hat{\underline{n}} = 0$ or $\underline{r} \cdot \underline{n} = \underline{a} \cdot \underline{n}$ $\quad$ distance to point $(\alpha, \beta, \gamma) = \left| \frac{a\alpha + b\beta + c\gamma + d}{\sqrt{a^2 + b^2 + c^2}} \right|$

$ax + by + cz + d = 0$ $\quad \left[\underline{n} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \right]$

Equation of a sphere: $|\underline{r} - \underline{c}|^2 = (\underline{r} - \underline{c}) \cdot (\underline{r} - \underline{c}) = a^2$

where c is the position vector of the centre and a = radius

Reciprocal Vectors

The two sets $\underline{a}, \underline{b}, \underline{c}$ and $\underline{a}', \underline{b}', \underline{c}'$ are called reciprocal sets if:

$\underline{a} \cdot \underline{a}' = \underline{b} \cdot \underline{b}' = \underline{c} \cdot \underline{c}' = 1$ and $a' \cdot b = a' \cdot c = b' \cdot a = \cdots = 0$

These reciprocal vectors are given by: (only exist if $\underline{a}, \underline{b}$, and $\underline{c}$ are not coplanar)

$$a' = \frac{\underline{b} \times \underline{c}}{\underline{a} \cdot (\underline{b} \times \underline{c})}, \quad b' = \frac{\underline{c} \times \underline{a}}{\underline{a} \cdot (\underline{b} \times \underline{c})}, \quad c' = \frac{\underline{a} \times \underline{b}}{\underline{a} \cdot (\underline{b} \times \underline{c})},$$

Index Notation

$\varepsilon_{ijk} \varepsilon_{ilm} = \delta_{jl} \delta_{km} - \delta_{jm} \delta_{kl}$ $\quad \varepsilon_{ijk} \varepsilon_{ijm} = 2\delta_{km}$ $\quad \varepsilon_{ijk} \varepsilon_{ijk} = 6$

$\varepsilon_{ijk} a_j a_k = \underline{a} \times \underline{a} = \underline{0}$

Vector spaces

A set of objects (vectors) $\underline{a}, \underline{b}, \underline{c}, \ldots$ is said to be a linear vector space if:

i) the set is closed under commutative and associative addition, so that:

$$\underline{a} + \underline{b} = \underline{b} + \underline{a}$$

$$(\underline{a} + \underline{b}) + \underline{c} = \underline{a} + (\underline{b} + \underline{c})$$

ii) the set is closed under multiplication by a scalar (any complex number) to form a new vector $\lambda \underline{a} \in V$, the operation being both distributive and associative, so that:

$$\lambda(\underline{a} + \underline{b}) = \lambda \underline{a} + \lambda \underline{b}$$

① $$(\lambda + \mu)\underline{a} = \lambda \underline{a} + \mu \underline{a}$$ where λ & μ are scalars

$$\lambda(\mu \underline{a}) = (\lambda \mu)\underline{a}$$

iii) there exists a null vector $\underline{0}$, such that $\underline{a} + \underline{0} = \underline{a}$ for all $\underline{a}$

iv) multiplication by unity leaves any vector unchanged such that $1 \times \underline{a} = \underline{a}$

v) all vectors have a corresponding negative vector (or additive inverse) such that $\underline{a} + (-\underline{a}) = \underline{0}$ (or $\underline{a} + \underline{a}' = \underline{0}$)

↳ it follows from ① with $\lambda = 1$, $\mu = -1$ that $\underline{a}' = -\underline{a} = -1 \times \underline{a}$

Note: if we restrict all vectors to be real then we obtain a real vector space, otherwise in general we obtain a complex vector space

The span of the set of vectors $\underline{a}, \underline{b}, \ldots, \underline{c}$ is defined to be the set of all vectors that may be written as a linear sum of the original vectors.

Linear Independence, Basis, Dimension

A set of vectors $\{\underline{v}_1, \ldots, \underline{v}_n\}$ is said to be linearly independent (abbreviated LI) if the only solution to the equation $\alpha_1 \underline{v}_1 + \alpha_2 \underline{v}_2 + \ldots + \alpha_n \underline{v}_n = \underline{0}$ is if all the scalar coefficients $\alpha_i = 0$. Otherwise the set is (LD). In an LD set at least one vector is redendant, since it can be represented as a linear sum of the others. You can test linear dependence by forming a matrix of the vectors and finding its determinant. If the determinant is zero then they are LD. $\text{span}(\underline{v}_1, \ldots, \underline{v}_m) = \{\alpha_1 \underline{v}_1 + \cdots + \alpha_m \underline{v}_m \mid \alpha_1, \ldots \alpha_m \in \mathbb{F}\}$.

A list of vectors $\underline{e}_1, \underline{e}_2, \ldots \underline{e}_n$ forms a basis for the space V if the elements of the list are LI and span V. Then any $\underline{a} \in V$ can be written as $\underline{a} = \alpha_i \underline{e}_i$ and the coefficients $(\alpha_1, \ldots, \alpha_n)$ for this form are known as the components or coordinates of $\underline{a}$ with respect to the basis.

Exchange Lemma: Number of basis vectors is equal to the dimension of V.

↳ actually: if there are n basis elements, and you have a set of m elements of V with m > n, then the set is LD

~~Linear Operators / Linear Maps~~ Some Useful Inequalities

Schwarz's Inequality: $|\langle \underline{a}, \underline{b} \rangle| \leq |\underline{a}||\underline{b}|$ ~~iff $\underline{a} = \lambda \underline{b}$~~ [~~To prove do $\langle d, d \rangle$ where $d = \alpha + \beta b$~~]

Pythagoras: if $\langle a, b \rangle = 0$ then $|\underline{a} + \underline{b}|^2 = |a|^2 + |b|^2$

Parallelogram Law: $|\underline{a} + \underline{b}|^2 + |\underline{a} - \underline{b}|^2 = 2(|\underline{a}|^2 + |\underline{b}|^2)$ [To prove expand LHS]

Triangle inequality: $|\underline{a} + \underline{b}| \leq |\underline{a}| + |\underline{b}|$ [Prove from Schwarz's inequality]

Linear Operators / Linear Maps and Matrices

A function $f: X \to Y$ from set X to set Y, known as the domain and the co-domain respectively, is a mapping. The image of f

$\operatorname{Im} f \equiv \{f(x) \mid x \in X\} \subseteq Y$.

f is: One-to-One (injective) if each $y \in Y$ is mapped to by at most one $x \in X$.

Onto (surjective) if each $y \in Y$ is mapped to by at least one $x \in X$

bijective ⟺ invertible → bijective if each $y \in Y$ is mapped to by precisely one element $x \in X$

Identity map: $id_X : X \to X$

Map composition: Given $f: X \to Y$, $g: Y \to Z$ we define their composition $g \circ f : X \to Z$ as the new map $(g \circ f)(x) = g(f(x))$ obtained by applying f first, then g.

singular: no inverse exists
non-singular: inverse exists

A map $g: Y \to X$ is the inverse of $f: X \to Y$ if $f \circ g = id_Y$ and $g \circ f = id_X$. If it exists this inverse mapping is usually written f^{-1}.

We focus on maps $f: V \to W$ whose domain V and codomain W are vector spaces, possibly of different dimensions.

A map f is linear if for all vectors $v_1, v_2 \in V$ and all scalars $\alpha \in \mathcal{F}$

↳ $f(v_1 + v_2) = f(v_1) + f(v_2)$

$f(\alpha v) = \alpha f(v)$

Matrices: Any $n \times m$ matrix is a linear map from $\mathcal{F}^m$ to $\mathcal{F}^n$.

For any $\underline{u}, \underline{v} \in \mathcal{F}^m$ we have that $A(\underline{u} + \underline{v}) = A(\underline{u}) + A(\underline{v})$ and $A(\alpha \underline{v}) = \alpha (A \underline{v})$

if $f: X \to Y$, $y_i = \sum_{j=1}^{n} A_{ij} x_j$

↳ $g \circ f = BA$ where A is the matrix corres. to f and B is the matrix to g

Null map/operator: $O\underline{x} = \underline{0}$ for all $\underline{x}$, Identity map $= id_X = I\underline{x}$

Matrix addition is commutative and associative. Matrix multiplication by scalar is distributive and associative

Matrix multiplication by matrix is associative, not commutative and distributive across addition

$\varepsilon_{ijk} a_j a_k = \underline{a} \times \underline{a} = \underline{0}$

Coordinate Maps: Given a vector space V over a field $\mathcal{F}$ with basis $e_1, \dots, e_n$ we have that any vector in V can be expressed as $v = \sum_{i=1}^{n} \alpha_i e_i$ where the α_i are the coordinates of the vector wrt the e_i basis. Introduce a mapping $f: \mathcal{F}^n \to V$ defined by $f\begin{pmatrix}\alpha_1 \\ \vdots \\ \alpha_n\end{pmatrix} = \sum_{i=1}^{n} \alpha_i e_i = v$, this is called a coordinate map. This is useful as $\alpha_i = (f^{-1}(v))_i$

Kernel: the set of all elements $v \in V$ for which $f(v) = 0$

Rank: the rank of f is the dimension of the image

↳ $f(0) = 0 \therefore 0 \in \ker f$

$\ker f$ is a vector subspace of V

$\operatorname{Im} f$ is a vector subspace of W

f surjective $\Leftrightarrow \operatorname{Im} f = W \Leftrightarrow \dim \operatorname{Im} f = \dim W$

f injective $\Leftrightarrow \ker f = \{0\} \Leftrightarrow \dim \ker f = 0$

The dimension Theorem: for $f: V \to W$, $\dim \ker f + \dim \operatorname{Im} f = \dim V$

↳ if f has an inverse (bijective) then its inverse f^{-1} is also a linea map, and $\dim V = \dim W$

if $\dim V = \dim W$ then f is bijective $\Leftrightarrow \dim \ker = 0 \Leftrightarrow \operatorname{rank} f = \dim W$

Recall that $f: V \to W$ is invertible iff $\dim W = \dim \operatorname{Im} f = \dim V$. An $n \times m$ matrix ~~as~~ A is a map from $\mathcal{F}^m$ to $\mathcal{F}^n$. So A^{-1} only exists if $m = n$ and $\operatorname{rank} A = n$. Other properties of the inverse:

↳ $(AB)^{-1} = B^{-1}A^{-1}$, $(A^{-1})^{-1} = A$, $(A^T)^{-1} = (A^{-1})^T$, $AA^{-1} = A^{-1}A = I_n$

Change of basis of the matrix representing a linear map:

A represents $f: V \to W$ with basis $v_1, \dots, v_m$ in V and $w_1, \dots, w_n$ in W. If we introduce new basises $v_1', \dots, v_m'$ and $w_1', \dots, w_n'$, the matrix representing the map is A'. (include φ as coordinate map for $v_1, \dots, v_m$ basis and ψ for $w_1, \dots, w_m$)

$A' = QAP^{-1}$ where P 'primes' V coordinates and Q primes W coordinates

$P^{-1} = \varphi^{-1} \circ \varphi'$, $P = (\varphi')^{-1} \circ \varphi$, $Q = (\psi')^{-1} \circ \psi$ [The most common use of this is for $f: V \to V$ so $Q = P$ and this is just for basis change]

We can also write $v_j = \sum_i P_{ij} v_i' \Leftrightarrow v_j' = \sum_i (P^{-1})_{ij} v_i$ (remembering that the vs are basis vectors)

↳ for $f: V \to V$

Solving Systems of Linear Equations

Suppose we have n simultaneous eqns in m unknowns:

$$A_{11}x_1 + \cdots + A_{1m}x_m = b_1$$
$$\vdots$$
$$A_{n1}x_1 + \cdots + A_{nm}x_m = b_n$$

This can be represented by the matrix equation $A\underline{x} = \underline{b}$ where A is an $n \times m$ matrix, $\underline{x}$ is an m-dimensional column vector, and $\underline{b}$ is an n-dimensional column vector.

If $\underline{b} = \underline{0}$ the system is called homogenous.

Let $\underline{x}_1$ be any one vector for which $A\underline{x}_1 = \underline{b}$. If such an $\underline{x}_1$ exists then the full space of solutions to $A\underline{x} = \underline{b}$ is the set $\underline{x} \in \{\underline{x}_0 + \underline{x}_1 \mid \underline{x}_0 \in \ker A\}$. So to find solutions, we first solve the homogenous equation $A\underline{x} = \underline{0}$ [to find $\underline{x}_0$s]. To these solutions we add a particular solution, $\underline{x}_1$ for $A\underline{x}_1 = \underline{b}$

Row rank equals column rank: we may view an $n \times m$ matrix A as a list of n row vectors $\underline{A_i} = (A_{i1}, \ldots, A_{im})$ or m column vectors, $\underline{A^i} = (A_{1i}, \ldots, A_{ni})$. Substituting $x_j = \delta_{kj}$ (standard orthogonal basis vectors) into the expression $A\underline{x}$ produces the k^{th} column vector $\underline{A^k}$. Therefore, $\operatorname{Im} A = \operatorname{span}(\underline{A^1}, \ldots, \underline{A^m})$. And viewed as a mapping, the rank of the matrix rank = no. LI column vectors. This is called the 'column rank' of the matrix A.

* rank = no. LI column vectors = no. LI row vectors where the no. LI column vectors is sometimes known as 'row rank'.

↳ This can be proved with The Dimension Theorem and The Orthogonal component Theorem.

Orthogonal component theorem: If W is a vector subspace of $\mathcal{F}^n$ and $W^\perp$ is its orthogonal complement then $\dim W + \dim W^\perp = n$

where $W^\perp \equiv \{\underline{v} \mid \underline{v} \cdot \underline{w} = 0, \text{ for all } \underline{w} \in W, \underline{v} \in \mathcal{F}^n\}$

↳ This is important as it means we can use row reduction which is clearer than manipulating columns.

Calculating the rank: row reduction Suppose we have a list of vectors $\underline{v}_1, \ldots, \underline{v}_n$. The space spanned by the list is unchanged under the following operations:

"row reduction operations"
- i) Swap any pair $\underline{v_i}$ and $\underline{v_j}$
- ii) Multiply any $\underline{v_i}$ by a nonzero scalar
- iii) Replace $\underline{v_i}$ by $\underline{v_i} + \alpha\underline{v_j}$

(This list of vectors can be rows or columns of a Matrix)

Row reduction: apply operations i) - iii) above to put matrix into "echelon form"

Echelon form: Index of first non-zero element of row $j+1$ > index of first of row j

↳ eg. $\begin{pmatrix} 1 & 0 & 3 \\ 0 & 3 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ & $\begin{pmatrix} 0 & 0 & 2 & 1 & 0 & 2 \\ 0 & 0 & 0 & -5 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 3 \end{pmatrix}$ are in echelon form [The line is just an indicator. The line can step down by at most one]

↳ obviously $\boxed{\text{rank} = 2}$ ↳ obviously $\boxed{\text{rank} = 3}$ ⇒ see line with *

echelon forms makes rank obvious

Note: Each row reduction operation has a corresponding "elementary matrix". Elementary Matrices are invertible to another elementary matrix.

↳ eg $\begin{pmatrix}1&0&0\\0&1&0\\\alpha&0&1\end{pmatrix}$ adds ($\alpha \times$ row 1) to row 3

Example of row reduction to echelon form:

$$\begin{pmatrix}0&1&-1\\2&3&-2\\2&1&0\end{pmatrix} \xrightarrow{\text{Swap } R1 \,\&\, R2} \begin{pmatrix}2&3&-2\\0&1&-1\\2&1&0\end{pmatrix} \xrightarrow{R3 = R3 - R1} \begin{pmatrix}2&3&-2\\0&1&-1\\0&-2&2\end{pmatrix} \xrightarrow{R_3 = R_3 - 2R_2} \begin{pmatrix}2&3&-2\\0&1&-1\\0&0&0\end{pmatrix}$$

Gaussian Elimination: Finding $\underline{x}$ in $A\underline{x} = \underline{b}$ by reducing the equation to echelon form by applying row reduction operations and then backsubstituting

Easiest to learn with an EXAMPLE:

$$\begin{pmatrix}1&1&2\\2&-1&10\\1&-2&3\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \begin{pmatrix}1\\4\\0\end{pmatrix}$$

Augmented matrix: $A' = \left(\begin{array}{ccc|c}1&1&2&1\\2&-1&10&4\\1&-2&3&0\end{array}\right)$

$R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - R_1$

$$= \left(\begin{array}{ccc|c}1&1&2&1\\0&-3&6&2\\0&-3&1&-1\end{array}\right)$$

$R_3 \to R_3 - R_2$

$$= \left(\begin{array}{ccc|c}1&1&2&1\\0&-3&6&2\\0&0&-5&-3\end{array}\right)$$

i.e $\begin{pmatrix}1&1&2\\0&-3&6\\0&0&-5\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \begin{pmatrix}1\\2\\-3\end{pmatrix}$

$\boxed{x_3 = 3/5}$, $-3x_2 + 6\left(\frac{3}{5}\right) = 2$, $\boxed{x_2 = \frac{8}{15}}$, $x_1 + \frac{8}{15} + 2\left(\frac{3}{5}\right) = 1$, $\boxed{x_1 = \frac{-11}{15}}$

EXAMPLE 2:

$$\begin{pmatrix}0&1&-1\\2&3&-2\\2&1&0\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \begin{pmatrix}1\\1\\b\end{pmatrix}$$

solution only exists if $b = -1$

upper 2x2 matrix

$$\vdots$$

$$\begin{pmatrix}2&3&-2\\0&1&-1\\0&0&0\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \begin{pmatrix}1\\1\\b+1\end{pmatrix}$$

↳ can't find x_3?! This happens with matrices without inverse (bottom row is all zeros) so strategy:

- Turn upper 2x2 matrix into a diagonal one (by applying $R_1 \to R_1 - 3R_2$ in this case and Let $x_3 = t$

↳ $\begin{pmatrix}2&0&1\\0&1&-1\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \begin{pmatrix}-2\\1\end{pmatrix}$

got rid of 3rd row as it is no use

↳ $2x_1 + t = -2$, $x_2 - t = 1$

↳ $\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \begin{pmatrix}-1\\1\\0\end{pmatrix} + t\begin{pmatrix}-1/2\\1\\1\end{pmatrix}$

Finding the inverse of a matrix

To find the inverse of a square matrix A, apply elementary row reduction operations to reduce $A \to I$ while simultaneously applying the same operations to the identity I. Then $(E_m \cdots E_1)A = I$ (where E_i = elementary matrix of the different row reduction operations) so $A^{-1} = (E_m \cdots E_1)I$. ∴ A^{-1} = the outcome of I with the operations applied.

eg. $A = \begin{pmatrix}1&1&2\\2&-1&10\\1&-2&3\end{pmatrix} \to \left(\begin{array}{ccc|ccc}1&1&2&1&0&0\\2&-1&10&0&1&0\\1&-2&3&0&0&1\end{array}\right) \to \left(\begin{array}{ccc|ccc}1&1&2&1&0&0\\0&-3&6&-2&1&0\\0&0&-5&1&-1&1\end{array}\right) \to \left(\begin{array}{ccc|ccc}1&0&0&\frac{17}{15}&-\frac{7}{15}&\frac{12}{15}\\0&1&0&\frac{4}{15}&\frac{1}{15}&-\frac{6}{15}\\0&0&1&-\frac{1}{5}&\frac{1}{5}&-\frac{1}{5}\end{array}\right)$

Augmented matrix (easier to go to echelon form first)

∴ $A^{-1} = \frac{1}{15}\begin{pmatrix}17&-7&12\\4&1&-6\\-1&1&-1\end{pmatrix}$

Don't do this to solve $A\underline{x} = \underline{b}$ (use Gaussian Elimination above) as A^{-1} might not exist and this takes twice as long

Determinant and trace of linear $V^n \to V^n$ maps

Multilinear maps: Suppose $V_1, \ldots, V_k$ are vector spaces over a common field of scalars $\mathcal{F}$. A map $f: V_1 \times \cdots \times V_k \to \mathcal{F}$ is multilinear (specifically k-linear) if it is linear in each variable separately:

$$f(\underline{v}_1, \ldots, \alpha \underline{v}_i + \alpha' \underline{v}_i', \ldots, \underline{v}_k) = \alpha f(\underline{v}_1, \ldots, \underline{v}_i, \ldots, \underline{v}_k) + \alpha' f(\underline{v}_1, \ldots, \underline{v}_i', \ldots, \underline{v}_k)$$

For the special case $k = 2$, the map is called bilinear. The dot product of two real vectors is an example of a bilinear map.

A multilinear map is alternating if it returns zero whenever two of its arguments are equal: $f(\underline{v}_1, \dots, \underline{v}_i, \dots, \underline{v}_i, \dots, \underline{v}_k) = 0$

The output of a multilinear ~~map~~ alternating map changes sign when two of its arguments are exchanged. [to prove expand $f(\underline{u}+\underline{v}, \underline{u}+\underline{v}) = 0$] $f(\underline{u}, \underline{v}) = -f(\underline{v}, \underline{u})$

Determinants definition

Note: For $A = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$, $\det A = |A| = a_1(b_2c_3 - b_3c_2) + a_2(b_3c_1 - b_1c_3) + a_3(b_1c_2 - b_2c_1)$

From this it is clear that $|A| = \underline{a} \cdot (\underline{b} \times \underline{c})$ = volume of parallelepiped

det A is a change in (oriented) volume of map A

↳ This also works for column vectors as $|A| = |A^T|$ below (this is just a useful way of thinking about it)

fancy definition ↓ Also true for rows from $|A| = |A^T|$ below

The determinant is the unique mapping from ~~nxn~~ $n \times n$ matrices to scalars that is n-linear alternating in the columns and takes the value 1 for the identity matrix.

Some immediate consequences of this definition:

Also apply to rows from $|A| = |A^T|$ below

- ↳ If two columns of A are identical then $\det A = 0$ (or if the column vectors are LD)
- Swapping two columns of A changes the sign of $\det A$
- If B is obtained from A by multiplying a single column of A by a factor ~~c~~ c then ~~...~~ $\det B = c \det A$
- If one column of A consists entirely of zeros then $\det A = 0$
- Adding a multiple of one column to another doesn't change $\det A$

Permutations of a List

A permutation of the list $(1, 2, \dots, m)$ is another list that contains each of the ~~elements~~ numbers $1, 2, \dots, m$ exactly once. In other words, it is a straightforwards shuffling of the order of the elements. There are $m!$ permutations of an m-element list.

Given a permutation P, we write $P(1)$ for the first element in the shuffled list, $P(2)$ for the second... etc. Then P can be written as $P = (P(1), P(2), \dots, P(m))$ or as $P = \begin{pmatrix} 1 & 2 & \dots & m \\ P(1) & P(2) & \dots & P(m) \end{pmatrix}$ which emphasises that P is a mapping from the top row to itself. From Any two permutation mappings we can ~~define~~ compose a new one PQ defined through $(PQ)(i) = P(Q(i))$. There is an identity mapping for which $P(i) = i$ and every P has an inverse $P^{-1} = \begin{pmatrix} P(1) & P(2) & \dots & P(m) \\ 1 & 2 & \dots & m \end{pmatrix}$.

identity permutation is even

Any Permutation P can be constructed from $(1, 2, \dots, m)$ by a sequence of pairwise element ~~...~~ exchanges. Even/odd permutations require an even/odd numbers of exchanges. The sign of P is defined as $\operatorname{sgn}(P) = \begin{cases} +1 & \text{if } P \text{ is an even permutation of } (1, 2, \dots, m) \\ -1 & \text{if } P \text{ is an odd permutation of } (1, 2, \dots, m) \end{cases}$

Given two permutations, P and Q: $\operatorname{sgn}(PQ) = \operatorname{sgn}(P)\operatorname{sgn}(Q)$, $\operatorname{sgn}(P^{-1}) = \operatorname{sgn}(P)$

Leibniz's Expansion of the Determinant

We can express each column vector $\underline{A}^j$ of the matrix A as a linear combination $\underline{A}^j = \sum_i A_{ij}\underline{e}_i$ of the column basis vectors $\underline{e}_1 = (1,0,0,\dots)^T, \dots, \underline{e}_n = (0,\dots,0,1)^T$

For any K-linear map δ we have that, by definition

$$\delta(\underline{A}^1, \dots, \underline{A}^k) = \delta\left(\sum_{i_1=1}^{n} A_{i_1,1}\underline{e}_{i_1}, \dots, \sum_{i_k=1}^{n} A_{i_k,k}\underline{e}_{i_k}\right)$$
$$= \sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\dots\sum_{i_k=1}^{n}(A_{i_1,1}\times\dots\times A_{i_k,k})\,\delta(\underline{e}_{i_1}, \dots, \underline{e}_{i_k})$$

Imposing the condition that δ be alternating means that the $\delta(e_{i_1}, \dots, e_{i_k})$ vanishes if two or more of the i_k are equal. Therefore we need consider only those $(i_1, \dots, i_n)$ that are permutations P of $(1, \dots, n)$.

As δ is alternating, we know that it changes sign under pairwise exchanges.

$$\therefore\ \delta(\underline{e}_{P(1)}, \dots, \underline{e}_{P(n)}) = \mathrm{sgn}(P)\,\delta(\underline{e}_1, \dots, \underline{e}_n)$$

Finally the condition that $\det I = 1$ sets $\delta(\underline{e}_1, \dots, \underline{e}_n) = 1$

↳ we know that $\delta(\underline{v}_1, \dots, \underline{v}_m)$ gives us the determinant as we have set it as a K-linear, alternating map and that it takes the value 1 for the identity matrix and by the definition of the determinant, we know that a mapping that fulfills these is uniquely the determinant.

The result is that $\det A = \sum_P \mathrm{sgn}(P)\, A_{P(1),1} A_{P(2),2} A_{P(3),3} \cdots A_{P(n),n}$

eg. $n = 2$, $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$, $P_1 = (1,2)$, $P_2 = (2,1)$, $\mathrm{sgn}\,P_1 = +1$, $\mathrm{sgn}\,P_2 = -1$

$$\therefore \det A = +1(A_{11}\times A_{22}) + -1(A_{21}A_{12})$$
$$\det A = A_{11}A_{22} - A_{21}A_{12}$$

Some Properties of Determinants

$\det A = \det A^T$, $\det(AB) = \det(A)\det(B)$

A is invertible iff $\det A \neq 0$ $\det(A^{-1}) = 1/\det A$

The determinant of a matrix that represents a linear map is independent of the basis used.

The A-level one → **Laplace's Expansion of the Determinant**

Given an $n\times n$ matrix A, let $A_{(i,j)}$ be the $(n-1)\times(n-1)$ matrix obtained by omitting the i^{th} row and j^{th} column of A. Define the cofactor matrix as:

$$c_{ij} = (-1)^{i+j}\det(A_{(i,j)}) \quad (= C(\text{minors}))$$

Its transpose is known as the adjugate matrix / classical adjoint of A:

$$c_{ij}^T = (\mathrm{adj}\,A)_{ji} = (-1)^{i+j}\det(A_{(i,j)}) \quad (= C(\text{minors})^T)$$

Just accept that: $\delta_{ij} \det A = \sum_{k=1}^{n} A_{ik} C_{jk} = \sum_{k=1}^{n} e_{ki} A_{kj}$

not very good for finding inverses

$$\Downarrow$$

$$(\det A) I = A \operatorname{adj} A = (\operatorname{adj} A) A \Rightarrow A^{-1} = \frac{1}{\det A} \operatorname{adj} A$$

Cramer's Rule: AKA solving $A\underline{x} = \underline{b}$ using determinants

Consider the set of simultaneous ~~equation~~ equations $A\underline{x} = \underline{b}$. For each $i = 1, \dots, n$, introduce a new matrix: $B_{(i)} = (\underline{A}^1, \dots, \underline{A}^{(i-1)}, \underline{b}, \underline{A}^{(i+1)}, \dots, \underline{A}^n)$ obtained by replacing the column i in A with $\underline{b}$. Note that this $\underline{b} = \sum_j x_j \underline{A}^j$. Then, using the multilinearity property of the determinant:

$$\begin{aligned}
\det(B_{(i)}) &= \det(\underline{A}^1, \dots, \underline{A}^{(i-1)}, \underline{b}, \underline{A}^{(i+1)}, \dots, \underline{A}^n) \\
&= \det(\underline{A}^1, \dots, \underline{A}^{(i-1)}, \textstyle\sum_j x_j \underline{A}^j, \underline{A}^{(i+1)}, \dots, \underline{A}^n) \\
&= \textstyle\sum_j x_j \det(\underline{A}^1, \dots, \underline{A}^{(i-1)}, \underline{A}^j, \underline{A}^{(i+1)}, \dots, \underline{A}^n) \\
&= \textstyle\sum_j x_j \delta_{ij} \det A \\
&= x_i \det A
\end{aligned}$$

$$x_i = \det(B_{(i)}) / \det A$$

eg: $\begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 4 & -3 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 9 \\ 8 \\ -2 \end{pmatrix}$

cramer's rule:
$x_1 = \det(B_{(1)}) / \det A$
$x_2 = \det(B_{(2)}) / \det A$
$x_3 = \det(B_{(3)}) / \det A$

$\det A = 1(+3) - 0(-4) + 2(-8)$
$= -13$

$B_{(1)} = \begin{pmatrix} 9 & 0 & 2 \\ 8 & 2 & 1 \\ -2 & -3 & 0 \end{pmatrix} \Rightarrow \det(B_{(1)}) = \dots = -13$

$B_{(2)} = \begin{pmatrix} 1 & 9 & 2 \\ 0 & 8 & 1 \\ 4 & -2 & 0 \end{pmatrix} \Rightarrow \det(B_{(2)}) = \dots = -26$

$B_{(3)} = \begin{pmatrix} 1 & 0 & 9 \\ 0 & 2 & 8 \\ 4 & -3 & -2 \end{pmatrix} \Rightarrow \det(B_{(3)}) = \dots = -52$

$\therefore \; x_1 = -13/-13 = 1$ $\qquad \underline{x} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$

$x_2 = -26/-13 = 2$

$x_3 = -52/-13 = 4$

Trace: The trace of an $n \times n$ matrix A is defined to be the sum of its diagonal elements: $\operatorname{tr} A \equiv \sum_{i=1}^{n} A_{ii}$

Trace is independent of basis

$$\downarrow$$

$$\operatorname{tr}(AB) = \operatorname{tr}(BA)$$

$$\operatorname{tr}(ABC) = \operatorname{tr}(CAB) = \operatorname{tr}(BCA)$$

Scalar Products

Orthonormal bases

An orthonormal basis for V is a set of basis vectors $\underline{e}_1, \dots, \underline{e}_n$ that satisfy: $\langle \underline{e}_i, \underline{e}_j \rangle = \delta_{ij}$. Any n orthonormal vectors in an n-dimensional inner-product space (a vector space with an additional structure of the inner product) form a basis:

Some important features:

- Coordinates of a vector: Let $\underline{v} = \sum_i \alpha_i \underline{e}_i$, then $\alpha_j = \langle \underline{e}_j, \underline{v} \rangle$
- Scalar product of $\underline{u} = \sum_i \alpha_i \underline{e}_i$ and $\underline{v} = \sum_j \beta_j \underline{e}_j$ is:

$$\langle \underline{u}, \underline{v} \rangle = \langle \textstyle\sum_i \alpha_i \underline{e}_i, \sum_j \beta_j \underline{e}_j \rangle = \sum_i \sum_j \alpha_i^* \beta_j \langle \underline{e}_i, \underline{e}_j \rangle = \sum_i \sum_j \alpha_i^* \beta_j \delta_{ij}$$

$$\langle \underline{u}, \underline{v} \rangle = \sum_i \alpha_i^* \beta_i \qquad = \sum_i \alpha_i^* \beta_i$$

- Matrix elements: Let $f: V \to V$ be a linear map.

  Applied to $\underline{v} = \sum_j \alpha_j \underline{e}_j$ we have $f(\underline{v}) = \sum_j \alpha_j f(\underline{e}_j)$.

  By coordinates of a vector, the i^{th} component of this is

  $$(f(\underline{v}))_i = \langle \underline{e}_i, \textstyle\sum_j \alpha_j f(\underline{e}_j) \rangle = \sum_j \langle \underline{e}_i, f(\underline{e}_j) \rangle \alpha_j$$

  comparing against $[A\underline{v}]_i = A_{ij}\alpha_j$ shows that the elements of the matrix A that represents f in this basis are: $A_{ij} = \langle \underline{e}_i, f(\underline{e}_j) \rangle$

  These are known as matrix elements of the map.

Every n-dimensional vector space V has an orthonormal basis: given any list of n LI vectors $\underline{v}_1, \dots, \underline{v}_n$ we can construct an orthonormal basis following the Gram-Schmidt procedure:

1. Start with $\underline{v}_1$. The first basis vector $\underline{e}_1$ is defined as:

$$\boxed{\underline{e}_1' = \underline{v}_1 \implies \underline{e}_1 = \frac{\underline{e}_1'}{|\underline{e}_1'|}}$$

2. Take $\underline{v}_2$. Subtract any component $\parallel$ to $\underline{e}_1$. Then normalise

$$\boxed{\underline{e}_2' = \underline{v}_2 - \langle \underline{e}_1, \underline{v}_2 \rangle \underline{e}_1 \implies \underline{e}_2 = \frac{\underline{e}_2'}{|\underline{e}_2'|}}$$

3. Similarly, take $\underline{v}_i$, subtracting any component $\parallel$ to $\underline{e}_1, \dots, \underline{e}_{i-1}$

$$\boxed{\underline{e}_i' = \underline{v}_i - \sum_{j=1}^{i-1} \langle \underline{e}_j, \underline{v}_i \rangle \underline{e}_j}$$

eg. $\underline{v}_1 = \begin{pmatrix}1\\1\\0\end{pmatrix}$, $\underline{v}_2 = \begin{pmatrix}2\\0\\1\end{pmatrix}$, $\underline{v}_3 = \begin{pmatrix}1\\-2\\-2\end{pmatrix}$

Can we find orthonormal basis with the same span?

$$\underline{e}_1' = \begin{pmatrix}1\\1\\0\end{pmatrix} \Rightarrow \boxed{\underline{e}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\end{pmatrix}}$$

$$\underline{e}_2' = \underline{v}_2 - \langle \underline{e}_1, \underline{v}_2 \rangle \underline{e}_1$$

$$= \begin{pmatrix}2\\0\\1\end{pmatrix} - \left(\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\end{pmatrix} \cdot \begin{pmatrix}2\\0\\1\end{pmatrix}\right) \times \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\end{pmatrix}$$

$$= \begin{pmatrix}1\\-1\\1\end{pmatrix} \Rightarrow \boxed{\underline{e}_2 = \frac{1}{\sqrt{3}}\begin{pmatrix}1\\-1\\1\end{pmatrix}}$$

$$\underline{e}_3' = \underline{v}_3 - \langle \underline{e}_2, \underline{v}_3 \rangle \underline{e}_2 - \langle \underline{e}_1, \underline{v}_3 \rangle \underline{e}_1$$

$$= \begin{pmatrix}1\\-2\\-2\end{pmatrix} - \left(\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\end{pmatrix} \cdot \begin{pmatrix}1\\-2\\-2\end{pmatrix}\right)\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\end{pmatrix} - \left(\frac{1}{\sqrt{3}}\begin{pmatrix}1\\-1\\1\end{pmatrix} \cdot \begin{pmatrix}1\\-2\\-2\end{pmatrix}\right)\frac{1}{\sqrt{3}}\begin{pmatrix}1\\-1\\1\end{pmatrix}$$

$$= \begin{pmatrix}1\\-1\\-2\end{pmatrix} \Rightarrow \boxed{\underline{e}_3 = \frac{1}{\sqrt{6}}\begin{pmatrix}1\\-1\\-2\end{pmatrix}}$$

Adjoint Maps

If f is a map then its adjoint $f^\dagger$ is the map that satisfies:

$$\langle f^\dagger(\underline{u}), \underline{v}\rangle = \langle \underline{u}, f(\underline{v})\rangle \quad \text{(equivalent to } \langle \underline{v}, f^\dagger(\underline{u})\rangle = \langle f(\underline{v}), \underline{u}\rangle)$$

$f^\dagger(\alpha_1 \underline{u}_1 + \alpha_2 \underline{u}_2) = \alpha_1 f^\dagger(\underline{u}_1) + \alpha_2 f^\dagger(\underline{u}_2)$: $f^\dagger$ is a linear map and is unique

Some properties:
- $(f^\dagger)^\dagger = f$
- $(f+g)^\dagger = f^\dagger + g^\dagger$
- $(\alpha f)^\dagger = \alpha^* f^\dagger$
- $(f\circ g)^\dagger = g^\dagger \circ f^\dagger$
- If f has an inverse then $(f^{-1})^\dagger = (f^\dagger)^{-1}$

Recall that the matrix representing f has elements $A_{ij} = \langle \underline{e}_i, f(\underline{e}_j)\rangle$

Similarly the matrix $f^\dagger$ has elements $\langle \underline{e}_i, f^\dagger(\underline{e}_j)\rangle = \langle f(\underline{e}_i), \underline{e}_j\rangle = \langle \underline{e}_j, f(\underline{e}_i)\rangle^* = A_{ji}^* = (A^\dagger)_{ij}$

So if f has a matrix A, then $f^\dagger$ is represented by $A^\dagger$ where $A^\dagger = (A^T)^*$

Hermitian, Unitary and normal maps

A map f is hermitian if it is self-adjoint : $f^\dagger = f$

Corresponding matrices have $A^\dagger = A$, if V is real then A is symmetric $A^T = A$.

The composition of two Hermitian maps is Hermitian iff they commute

↳ If $f = f^\dagger$ and $g = g^\dagger$ then $(f\circ g)^\dagger = (g\circ f)^\dagger = f\circ g = g\circ f$

iff $f\circ g = g\circ f$ (i.e. iff $[f,g]=0$)

↳ where $\underset{\uparrow \text{ commutator}}{[f,g]} = f\circ g - g\circ f$: If you apply two different maps in different orders you usually get different results, this measures how different

Normal maps satisfy $[f,f^\dagger]=0$

Hermitian and ~~normal~~ unitary maps are normal

A map f is unitary if it preserves the scalar product for all $\underline{u},\underline{v}\in V$

↳ $\langle \underline{u},\underline{v}\rangle = \langle f(\underline{u}), f(\underline{v})\rangle = \langle (f^\dagger\circ f)(\underline{u}), \underline{v}\rangle \Rightarrow \langle \underline{u}, (f\circ f^\dagger)(\underline{v})\rangle$

∴ So for unitary maps, $f^\dagger\circ f = f\circ f^\dagger = id_V$

and, the corresponding matrix U satisfies $\boxed{UU^\dagger = U^\dagger U = I_n}$ This means that $\langle \underline{U}^i, \underline{U}^j\rangle = \delta_{ij} = \langle \underline{U}_i, \underline{U}_j\rangle \Rightarrow$ columns and rows are both orthonormal. For real vector spaces unitary maps correspond to orthogonal matrices.

orthogonal = Unitary

Unitary maps are reflections and rotations, lets call such a map R.

Because R conserves scalar product, we have $\langle R\underline{u}, R\underline{v}\rangle = \langle \underline{u},\underline{v}\rangle$

↳ $= \langle R^T R\underline{u}, \underline{v}\rangle$ (in a real vector space) $\Rightarrow \langle \underline{u}, R^T R\underline{v}\rangle = \langle \underline{u},\underline{v}\rangle$

∴ $R^T R = I \Rightarrow \det(R^T R) = 1 \Rightarrow \det(R^T)\det(R) = 1 \Rightarrow \det(R)\det(R) = 1$

∴ $\det(R) = \pm 1$ If $\det R = +1$: Pure Rotation If $\det R = -1$: rotation & reflection

↳ Example: $R = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1\end{pmatrix}$ (Pure rotation)

~~Recall that trace is independent of basis. So if~~ we have $\det R = +1$ (Pure rotation) and $R^T R = I$ then: $\operatorname{tr} R = 1 + 2\cos\theta$, θ = rotation angle

Orthogonal Complement Theorem

Let $W \subset V$ be a vector subspace and define $W^\perp = \{\underline{v} \mid \langle \underline{w}, \underline{v} \rangle, \text{ all } \underline{w} \in W, \underline{v} \in V\}$

Then: $W^\perp$ is a vector subspace of V

$W \cap W^\perp = \{\underline{0}\}$

$\dim W + \dim W^\perp = \dim V$

Dual Space

wrt. orthogonal basis: $\langle \underline{u}, \underline{v} \rangle = \sum_{i=1}^{n} u_i^* v_i = (u_1^* \dots u_n^*) \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$

Now consider $\langle \underline{u} | \cdot = \langle \underline{u}, \cdot \rangle$ = "bra u". This is the linear map between V & $\mathcal{F}$. Write $\underline{v} \in V$ as $| \underline{v} \rangle$ = "ket v". $\langle \underline{u} | \cdot$ is a dual vector.

The set of all such $\langle \underline{u} | \cdot$ on n-dim vector space V is itself another vector space. If $| \underline{u} \rangle$ is $\begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}$ then there is a corresponding $\langle \underline{u} | = (u_1^* \dots u_n^*)$ ie. $\alpha | a \rangle + \beta | b \rangle$ associated with $\alpha^* \langle \underline{a} | + \beta^* \langle \underline{b} |$. (every ket has a bra).

Eigenstuff

If $f(\underline{v}) = \lambda \underline{v}$ for some $\underline{v} \neq \underline{0}$ then $\underline{v}$ is an eigenvector of f with eigenvalue λ. (f scales $\underline{v}$ by factor λ). We can rewrite $f(\underline{v}) = \lambda \underline{v}$ as $(f - \lambda\, id_V)(\underline{v}) = \underline{0}$. This has nontrivial solutions $\underline{v}$ iff $\operatorname{rank}(f - id_V) < n$ (because of the dimension theorem). That is, if $\boxed{\det(f - \lambda\, id_V) = 0}$. The vector subspace $Eig_f(\lambda) = Ker(f - \lambda\, id_V)$ of V is called the eigenspace associated with the eigenvalue λ

Characteristic Polynomial

Choose a basis for V and let A be matrix that represents f. The characteristic polynomial of A is defined to be $\chi_A(\lambda) = \det(A - \lambda I) = \dots$

$\sum_Q \operatorname{sgn}(Q) (A - \lambda I)_{Q(1),1} \cdots (A - \lambda I)_{Q(n),n}$ ← Leibnitz's expansion of the determinant.

It is independent of basis because, in another basis the matrix that represents f is $A' = PAP^{-1}$ for which the characteristic polynomial is $\chi_{PAP^{-1}}(\lambda) = \det[P(A - \lambda I)P^{-1}] = \det(A - \lambda I) = \chi_A(\lambda)$

To find eigenvalues, solve the characteristic equation: $0 = \chi_A(\lambda) = \det(A - \lambda I)$ bearing in mind that any n^{th} order polynomial has n roots, possibly repeated. Eigenvalues are in general complex numbers, even for maps defined on a real vector space. In the rest of this section we'll assume a complex vector space.

After finding the eigenvalues λ_i, try to find the corresponding eigenvectors (If they exist) by trying to construct the eigenspace $Ker(A - \lambda_i\, id_V)$

$\det A = \prod_i \lambda_i$ $\qquad$ $tr\, A = \sum_i \lambda_i$

Eigenvectors for Hermitian matrices are orthogonal. Eigenvectors for rotation matrices are orthogonal.

If an $n \times n$ matrix has n orthogonal eigenvectors then these vectors are a basis, and in this new ~~matrix~~ basis the matrix is diagonal.

The eigenvalues of a Hermitian map are ~~orthogonal~~ real. The eigenvectors corresponding to distinct eigenvalues of a Hermitian map are orthogonal.
The eigenvalues of a unitary map are complex numbers with unit modulus.
The eigenvalues corresponding to distinct eigenvalues of a unitary map are orthogonal.

↳ Generalisation to normal maps: Let $f: V \to V$ be a normal linear map. (i.e $f \circ f^{+} = f^{+} \circ f$). If f has an eigenvector $\underline{v}$ with eigenvalue λ, then this $\underline{v}$ is an eigenvector of f^{+} with eigenvalue λ^*.

The eigenvectors of a normal map (in a complex vector space ; doesn't hold ~~the~~ true in a real vector space) form an orthogonal basis.

If f has an orthonormal basis of eigenvectors then it is normal.

Diagonalisation

A linear map $f: V \to V$ is diagonalisable iff there is a basis in which the matrix that represents f is diagonal.

A linear map $f: V \to V$ is diagonalisable iff the eigenvectors of f are a basis of V. Relative to this basis the matrix representing $A = \text{diag}(\lambda_1, \ldots, \lambda_n)$ where λ_i are the eigenvalues of f.

An $n \times n$ matrix A with entries $A_{ij} \in \mathcal{F}$ is diagonisable iff there is an invertible $n \times n$ map P with $P_{ij} \in \mathcal{F}$ such that $A = PDP^{-1}$ where D is some diagonal matrix.

An $n \times n$ matrix A with entries is $\mathcal{F}$ is diagonisable iff it has n eigenvectors $\underline{v}_1, \ldots, \underline{v}_n$ that form a basis of $\mathcal{F}^n$. In that case if we define the matrix $Q = (\underline{v}_1, \ldots, \underline{v}_n)$ whose i^{th} column contains the coordinate of $\underline{v}_i$, it follows that $Q^{-1} A Q = \text{diag}(\lambda_1, \ldots, \lambda_n)$ where λ_i = eigenvalue of $\underline{v}_i$.

Simultaneous diagonalisation: Let A and B be two ~~diagonal~~ diagonalisable matrices. There is a basis in which A and B are both diagonal iff $[A, B] = 0$ (they commute)